

Systematic construction of genuine multipartite entanglement criteria in continuous variable systems using uncertainty relations

F. Toscano,¹ A. Saboia,¹ A. T. Avelar,² and S. P. Walborn¹

¹*Instituto de Física, Universidade Federal do Rio de Janeiro,
Caixa Postal 68528, Rio de Janeiro, RJ 21941-972, Brazil*

²*Instituto de Física, Universidade Federal de Goiás,
Caixa Postal 131, Goiânia, GO 74001-970, Brazil*

A general procedure to construct criteria for identifying genuine multipartite continuous variable entanglement is presented. It relies on the definition of adequate global operators describing the multipartite system, the positive partial transpose criterion of separability, and quantum mechanical uncertainty relations. As a consequence, each criterion encountered consists in a single inequality nicely computable and experimentally feasible. Violation of the inequality is sufficient condition for genuine multipartite entanglement. Additionally we show that the previous work of van Loock and Furusawa [Phys. Rev. A, **67**, 052315 (2003)] is a special case of our result.

I. INTRODUCTION

Genuine multipartite entanglement – entanglement between three or more quantum systems – is essential to harness the full power of quantum computing in either the circuit [1] or one-way models [2] as well as to the security in multi-party quantum encryption protocols [3]. Additionally, it provides increased precision in quantum metrology [4, 5], furnishes the resource to solve the Byzantine agreement problem [6], and allows for multi-party quantum information protocols such as open destination quantum teleportation [7]. There is also evidence that it is responsible for efficient transport in biological systems [8], and is linked to fundamental aspects of phase transitions [9].

Experimental identification of genuine multipartite entanglement is essential, since in order to reliably realize any multipartite entanglement-based task, it is necessary to confirm the presence of genuine multipartite entangled states. Although quantum state tomography can provide all of the available information about the system, it requires a number of measurements that increases exponentially with the number of subsystems. Thus, entanglement witnesses composed of an abbreviated number of measurements are the most viable method for identifying genuine multipartite entanglement. This is true especially when the system is of high dimension, due to the dimension of the Hilbert space of each subsystem and/or the number of constituent subsystems.

In particular, for a continuous variable (CV) system composed of n subsystems or modes, quantum state tomography is not viable in general. This is rapidly becoming an important experimental concern, since possibly genuine multipartite CV entanglement has been produced for three degenerate [10] and non-degenerate modes [11], and more recently for a large number of temporal [12] or spectral [13, 14] modes. Even for the specific case of two-modes, the difficulty of state tomography has also led to several criteria involving second-order [15–18] and higher-order moments [19–24] of the canonical variables. Most of these criteria are based on the positive

partial transpose (PPT) argument [25, 26] and uncertainty relations involving the variance [15–18] or entropy [22, 23] of marginal distributions. Motivated by the impossibility of PT based entanglement criteria to detect bound entanglement [27], in [28] the authors construct bipartite entanglement witnesses solving the so-called separability eigenvalue equation. The authors extend their approach in [29] deriving a set of algebraic equations, which yield the construction of arbitrary multipartite entanglement witnesses using general Hermitian operators. This approach is powerful when several measurements of different Hermitian operators are considered in order to feed the optimisation problem involved to obtain the optimal entanglement criteria with the measurements at hand. Despite the lack of economy in terms of number of measurements, it was successfully implemented in [14] to characterise multipartite entanglement in multimode frequency-comb Gaussian states, where the entire covariance matrix of the state was measured [30].

From a practical point of view, genuine entanglement criteria that are economical in terms of the number of measurement required for their implementation are more desirable. A first step in this direction for CV systems was made by van Loock and Furusawa [31] using a variance criterion to test full n -partite inseparability. These criteria do not require reconstruction of the covariance matrix, and for this reason they were preferred to test full n -partite inseparability in most of the CV multipartite states generated in the laboratory [10, 11, 13, 32–34]. However, genuine multipartite entanglement is different from n -partite inseparability, which can be produced by mixing quantum states with fewer than n entangled subsystems. In general, to prove genuine multipartite entanglement, one must show that the state cannot be written as a convex sum of biseparable states [35, 36]. In reference [36], the authors show that one of the criterion of van Loock and Furusawa, consisting of a single inequality for a proper non-local operator, is indeed a genuine multipartite entanglement criterion, and extend the criterion also for the product of variances. They also show how to adapt some of the criteria in [31], for three and

four modes, in order to obtain genuine entanglement criteria. It is important to note that the criteria in [31, 36] are based on uncertainty relations (UR) only for the variances of non-local linear observables (NLO) (*i.e.* linear combination of canonical conjugate local observables).

In this work, we solve the problem of how to construct genuine entanglement criteria for n CV modes using the positive partial transposition (PPT) separability criterion in conjunction with general uncertainty relations for non-local linear observables (URNLO). We will call these “PPT+URNLO” criteria. Our systematic method consists of adequate definitions of global operators of the n -partite system, which can be employed together with a wide range of quantum mechanical uncertainty relations, producing classes of unique inequalities that test genuine n -partite entanglement. In particular, we derive the unique family constituting single pairs of global n -mode position and momentum operators that simultaneously test entanglement in all possible bipartitions and also genuine n -partite entanglement. In the multipartite scenario, the criteria of van Loock and Furusawa [31] and also the criteria in [36–38] constitute particular cases of our results. In the bipartite scenario, we recover the most popular entanglement criteria [15–18, 22, 23].

The versatility of our results is twofold: i) for a fixed type of uncertainty relation we give the recipe of how to search all the pairs of non-local operators that could certify genuine multipartite entanglement in a given prepared target state, ii) for a fixed pair or a set of pairs of non-local operators whose marginal distributions are available to the experimentalist, we provide the possibility to search which uncertainty relation is more suitable to certify genuine multipartite entanglement.

In addition to showing that the criteria in [16–18, 31, 36–38], originally derived using the Cauchy-Schwarz inequality can be rederived using PPT separability criterion explicitly, we provide a general framework to obtain genuine entanglement criteria using PPT arguments and general uncertainty relations for non-local linear observables. If we consider that most of the target states in quantum information tasks are pure and their experimental implementation are meant to approximate these states, mixed bound entangled states are practically excluded. Even for mixed states, bound entanglement is a rare phenomena [39], so genuine entanglement criteria based on PPT arguments which are economical for their implementation are very useful.

This work is organised as follows. In Section II we establish the general idea of the class of “PPT+URNLO” criteria and apply it to the case of two mode systems, and give several examples of well known bipartite entanglement criteria that belong to this class. In section III we set the notation to describe all the bipartitions of an n -mode system and we review the definition of genuine multipartite entanglement. Section IV is devoted to the derivation of “PPT+URNLO” entanglement criteria to test bipartite entanglement in the multipartite scenario. In section V we use these bipartite criteria to build sev-

eral criteria for genuine multipartite entanglement. Concluding remarks are given in section VII.

II. BIPARTITE CRITERIA FOR TWO-MODE SYSTEMS

Let’s us begin introducing our approach in the case of a bipartite system. For states $\hat{\rho}$ of two bosonic modes, several entanglement criteria have been developed to detect bipartite entanglement [15–19, 21–24, 28, 40, 41]. Most of these criteria [15–19, 21–24, 41] rely on the PPT separability criterion. We note that several of these criteria [16–18] were not originally derived using the PPT separability criterion explicitly. Nevertheless, in section II A we show that they are indeed special cases of a general PPT separability criterion.

Some of these entanglement criteria [15–18, 22–24] are constructed using quantum-mechanical uncertainty relations that are based on quantities associated with operators of the form

$$\hat{u} = h_1 \hat{x}_1 + h_2 \hat{x}_2 \quad (1a)$$

$$\hat{v} = g_1 \hat{p}_1 - g_2 \hat{p}_2, \quad (1b)$$

where h_j and g_j are arbitrary real numbers. We call these “non-local” observables in the sense that they are linear combinations of observables on both systems. In general, \hat{u} and \hat{v} do not commute: $[\hat{u}, \hat{v}] = i\gamma \hat{1}$, where $\gamma = h_1 g_1 - h_2 g_2$. Then, a generic URNLO will be satisfied by all states $\hat{\rho}$, and can be written as

$$F[\hat{\rho}, P_{\hat{u}}, P_{\hat{v}}] \geq f(|\gamma|), \quad (2)$$

where f is an increasing function of its argument and F is a functional, involving quantities that can be either directly measured or determined from the marginal probabilities distributions $P_{\hat{u}}(u) = \langle u | \hat{\rho} | u \rangle$ and $P_{\hat{v}}(v) = \langle v | \hat{\rho} | v \rangle$ that are associated with the measurement of \hat{u} and \hat{v} on the state $\hat{\rho}$.

Using uncertainty relations of this type, a PPT+URNLO bipartite entanglement criteria can be derived in the following generic way. A consequence of the PPT separability criterion [15, 25, 26] is that if the partial transpose operator $\hat{\rho}^T$ [49] does not correspond to a proper density operator (*i.e.* it has negative eigenvalues) then the original state is entangled with respect to that bipartition. This means that the URNLO

$$F[\hat{\rho}^T, P_{\hat{u}}, P_{\hat{v}}] \geq f(|\gamma|) \quad (3)$$

can be violated because \hat{u} and \hat{v} do not necessarily verify an uncertainty relation when the operator $\hat{\rho}^T$ does not correspond to a quantum state. This only occurs when the original bipartite state $\hat{\rho}$ is entangled. Violation of Eq.(3) is only a sufficient condition for bipartite entanglement, and is not a necessary one. Nevertheless, criteria of this type are appealing due to the reduced number

of measurements required, and so they are widely used in experimental detection of entanglement in continuous variable systems.

Partial transposition is a non-physical transformation. Thus, inequality (3) is a useful entanglement criterion only if there is a simple way to obtain $F[\hat{\rho}^T, P_{\hat{u}}(\xi), P_{\hat{v}}(\xi)]$ from measurements on the original state $\hat{\rho}$. This connection can be made by considering the fact that transposition is equivalent to a mirror reflection in phase space [15], taking $(x, p) \rightarrow (x, -p)$. For two mode states, non-local observables of the form

$$\hat{\mu} = h_1 \hat{x}_1 + h_2 \hat{x}_2 \quad (4a)$$

$$\hat{\nu} = g_1 \hat{p}_1 + g_2 \hat{p}_2 \quad (4b)$$

are then transformed as $\hat{\mu} \rightarrow \hat{u}$ and $\hat{\nu} \rightarrow \hat{v} = \pm g_1 \hat{p}_1 \mp g_2 \hat{p}_2$ where the plus (minus) sign corresponds to transposition on the second (first) mode. For this reason, throughout this paper we will refer to μ, ν as the “original” variables, and u, v as the “mirrored” variables.

With this correspondence between the original and mirrored variables, it can be shown that functionals related to probability distributions of μ and ν on the original state ρ are equivalent to those related to u and v on the partially transposed state:

$$F[\hat{\rho}^T, P_{\hat{u}}, P_{\hat{v}}] = F[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}]. \quad (5)$$

Note that in general $\hat{\mu}$ and $\hat{\nu}$ also do not commute: $[\hat{\mu}, \hat{\nu}] = i\delta\hat{1}$, where $\delta = h_1 g_1 + h_2 g_2$. Thus, they satisfy the equivalent uncertainty relation

$$F[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] \geq f(|\delta|). \quad (6)$$

Combining Eqs.(3), (5) and (6) we can write any PPT+URNLO bipartite entanglement criterion as:

$$F[\hat{\rho}^T, P_{\hat{u}}, P_{\hat{v}}] = F[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] \geq f(|\bar{\delta}|), \quad (7)$$

where $|\bar{\delta}| = \max\{|\gamma|, |\delta|\}$. Thus, for two mode systems we have $|\bar{\delta}| = |h_1 g_1| + |h_2 g_2|$. Due to the UR in Eq.(6) violation of the inequality in Eq.(7) is only possible in the cases when $|\bar{\delta}| = |\gamma| > |\delta|$ and implies entanglement. Thus, it is necessary to choose γ and δ adequately.

A. Examples of Bipartite Criteria

Let us briefly review some PPT+URNLO bipartite entanglement criteria that appear in the literature. In Table I we identify the correspondence between our notation and previous entanglement criteria.

In the case where F is the sum of variances $\Delta^2 \hat{\xi}$ of the marginal distributions $P_{\hat{\xi}}(\xi)$ ($\xi = \mu, \nu$), we have the sum of variance entanglement criterion:

$$F_{Lin}[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] \equiv \Delta^2 \hat{\mu} + \Delta^2 \hat{\nu} \geq f_{Lin}(|\bar{\delta}|) = |\bar{\delta}|, \quad (8)$$

that appears in Eq.(11) of Ref. [15] and in Eq.(3) of Ref. [16] (the non-local observables can be mapped in

our notation through the identification showed in Table I).

The product of variance entanglement criterion given in Eq.(6) of Ref. [17] and in Eq.(28) of Ref. [18] can be obtained if we choose the functional F as:

$$F_H[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] \equiv \Delta^2 \hat{\mu} \Delta^2 \hat{\nu} \geq f_H(|\bar{\delta}|) = 1/4|\bar{\delta}|^2. \quad (9)$$

(see Table I for the identification of the non-local observables in references [17, 18] within our notation). The Shannon-entropic entanglement criterion is obtained when the functional F is the sum of Shannon entropies:

$$\begin{aligned} F_E[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] &= h[P_{\hat{\mu}}] + h[P_{\hat{\nu}}] \geq \\ &\geq f_E(|\bar{\delta}|) = \ln(\pi e |\bar{\delta}|), \end{aligned} \quad (10)$$

$h[P] \equiv -\int dx P(x) \ln(P(x))$ is the Shannon entropy of the probability distribution function $P(x)$. This new criterion presented here is based on a new uncertainty relation, $h[P_{\hat{u}}] + h[P_{\hat{v}}] \geq \ln(\pi e |\langle [\hat{u}, \hat{v}] \rangle|)$, recently proved in Ref. [42] for generic operators \hat{u} and \hat{v} in an n -mode bosonic system that are generic linear combination of the position and momentum of each mode. In the particular case when the mirrored operators \hat{u}, \hat{v} form conjugate pairs, *i.e.* they are related by a $\pi/2$ rotation therefore the wavefunctions corresponding to the eigenstates $|u\rangle$ and $|v\rangle$ are related by a Fourier Transform, we recover the bipartite entanglement criteria in Eq.(13) of [22] that used the old entropic uncertainty relation for conjugate pairs of operators derived in [43] (see Table I for the identification of the non-local operators within our notation). Nevertheless, here we have extended the result in [22] because the Shannon-entropic entanglement criterion given in our Eq.(10) is valid for generic linear non-local operators $\hat{\mu}$ and $\hat{\nu}$ not necessarily conjugate pairs.

The three entanglement criterion given in Eqs.(8), (9) and (10) can be written all together in a single inequality:

$$\begin{aligned} \ln[\pi e(\Delta^2 \hat{\mu} + \Delta^2 \hat{\nu})] &\geq \ln(2\pi e \Delta \hat{\mu} \Delta \hat{\nu}) \geq \\ &\geq h[P_{\hat{\mu}}] + h[P_{\hat{\nu}}] \geq \ln(\pi e |\bar{\delta}|). \end{aligned} \quad (11)$$

In this way, we see that the Shannon-entropic entanglement criterion is the strongest criterion because its corresponding inequality can be violated in cases when the other two are not, thus it can detect bipartite entanglement when the other two do not. Examples of bipartite states whose entanglement can be detected with the Shannon-entropic entanglement criterion but can't be detected with either variance criteria were shown in Ref. [22].

The examples presented here show that in order to create PPT+URNLO bipartite entanglement criteria we only need URs of the form Eq. (2), valid for any pair \hat{u} and \hat{v} of non-commuting linear non-local operators. However, the most common UR relations are defined for a single bosonic mode. Nevertheless, it is straightforward to use this type of UR if we restrict ourselves to conjugate pairs \hat{u} and \hat{v} of non-local operators because these operators define a type of “non-local” mode of its own right.

| Reference | Equation | Criterion | Order | “Original” non-local position: $\hat{\mu} \equiv h_1\hat{x}_1 + h_2\hat{x}_2$ | “Original” non-local momentum: $\hat{\nu} \equiv g_1\hat{p}_1 + g_2\hat{p}_2$ |
|------------|----------|-----------|-------|----------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------|
| Simon [15] | Eq.(11) | F_{Lin} | 2nd | $h_1\hat{x}_1 = d_1\hat{q}_1 + d_2\hat{p}_1$ $h_2\hat{x}_2 = d_3\hat{q}_2 + d_4\hat{p}_2$ | $g_1\hat{p}_1 = d'_1\hat{q}_1 + d'_2\hat{p}_1$ $g_2\hat{p}_2 = d'_3\hat{q}_2 + d'_4\hat{p}_2$ |
| DGCZ [16] | Eq.(3) | F_{Lin} | 2nd | $h_1 = a , h_2 = \frac{1}{a}$ | $h_1 = a , g_2 = -\frac{1}{a}$, with real a |
| MGVT [17] | Eq.(6) | F_H | 2nd | $h_1 = 1, h_2 = 1$ | $g_1 = 1, g_2 = -1$ |
| GMVT [18] | Eq.(28) | F_H | 2nd | $h_1\hat{x}_1 = a_1\hat{q}_1 + a_3\hat{p}_1$ $h_2\hat{x}_2 = a_2\hat{q}_2 + a_4\hat{p}_2$ | $g_1\hat{p}_1 = b_1\hat{p}_1 + b_3\hat{q}_1$ $g_2\hat{p}_2 = b_2\hat{p}_2 + b_4\hat{q}_2$ |
| WTSTM [22] | Eq.(13) | F_E | > 2 | $h_1 = 1, h_2 = \pm 1$ | $g_1 = 1, g_2 = \mp 1$ |
| STW [23] | Eq.(16) | F_E | > 2 | $h_1 = 1, h_2 = \pm 1$ | $g_1 = 1, g_2 = \mp 1$ |

TABLE I: Popular bipartite entanglement criteria that are of the type we call PPT+URNLO. In all cases the mirrored operators are (see the text): the non-local position $\hat{u} = \hat{\mu}$ and the non-local momentums $\hat{v} = -g_1\hat{p}_1 + g_2\hat{p}_2$ (if the partial transposition is with respect of the first mode) or $\hat{v} = g_1\hat{p}_1 - g_2\hat{p}_2$ (if the partial transposition is with respect of the second mode). We indicate the order of the moments of the “original” operators that appear in each criterion. Note that tabulated operators on the l.h.s. are ours.

For example, the UR in terms of the Rényi entropies were used in Eq.(16) of [23] to create a PPT+URNLO bipartite entanglement criterion. We recall the UR in terms of the Rényi entropies: $h_\alpha[P] = \frac{1}{1-\alpha} \ln [\int dx P^\alpha(x)]$, derived in [44]:

$$h_\alpha[P_{\hat{\mu}}] + h_\beta[P_{\hat{\nu}}] \geq -\frac{1}{2(1-\alpha)} \ln \frac{\alpha}{\pi|\gamma|} - \frac{1}{2(1-\beta)} \ln \frac{\beta}{\pi|\gamma|}, \quad (12)$$

that is valid for any conjugate pairs with $[\hat{u}, \hat{v}] = i\gamma\hat{1}$ and $1/\alpha + 1/\beta = 2$.

III. DEFINITION OF GENUINE MULTIPARTITE ENTANGLEMENT

The same concepts outlined in the last section can be used to develop entanglement criteria for multipartite systems. A multipartite system consisting of n parts, can be entangled in a number of different ways. For example, an n -partite state is said to be *genuinely* n -partite entangled if it cannot be prepared by mixing states that are separable with respect to some bipartition, dividing the constituent sub-systems into two groups. Thus, in order to define genuine multipartite entanglement in an n -mode state $\hat{\rho} \in \otimes_{i=1}^n \mathcal{H}_i$, we need to fix the notation to specify the different possible bipartitions of the system. We denote a bipartition of the system by $\vec{\alpha}|\vec{\beta}$, where the vectors of integer indexes $\vec{\alpha} \equiv (\alpha_1, \dots, \alpha_{n_A})$ and $\vec{\beta} \equiv (\beta_1, \dots, \beta_{n_B})$ indicate the modes belonging to each part, and α_i, β_i are integers in the set $\{1, \dots, n\}$. Thus, part A of the bipartition denoted by $\vec{\alpha}$, has n_A modes and part B denoted by $\vec{\beta}$ has $n_B = n - n_A$ modes. For convenience we order $\alpha_i < \alpha_{i+1}$ ($i = 1, \dots, n_A$) and $\beta_j < \beta_{j+1}$ ($j = 1, \dots, n_B$).

The different bipartitions of an n mode system can be classified according to the number of modes in each part. Thus, the class (n_A, n_B) contains all possible bipartitions with n_A modes in part A and n_B modes in part B . We

| n (modes) | n_A^{max} (classes) | L (partitions) |
|-------------|-------------------------------------------|-------------------|
| 2 | 1 | 1 |
| 3 | 1 | 3 |
| 4 | 2 | 7 |
| 5 | 2 | 15 |
| 6 | 3 | 31 |
| \vdots | \vdots | \vdots |
| n | $n/2$ for n even, $(n-1)/2$ for n odd | $L = 2^{n-1} - 1$ |

TABLE II: Number of modes, classes, and partitions.

call n_A^{max} the number of different bipartition classes, *i.e.* $n_A = 1, \dots, n_A^{max}$ where $n_A^{max} = n/2$ if n is even and $n_A^{max} = (n-1)/2$ if n is odd. In a given class (n_A, n_B) there are N_{n_A} bipartitions that correspond to different labels $\vec{\alpha}|\vec{\beta}$. For $n_A \neq n/2$, $N_{n_A} = \binom{n}{n_A}$, and for $n_A = n/2$, $N_{n_A} = \frac{1}{2} \binom{n}{n_A}$. It is easy to see that for either n odd or even, there are a total of $L = 2^{n-1} - 1$ different bipartitions. For example, in the case of a four mode system ($n = 4$), we have two classes ($n_A^{max} = 2$): i) $(n_A = 1, n_B = 3)$ that corresponds to the $N_{n_B} = 4$ bipartitions, $\vec{\alpha}|\vec{\beta} = 1|234, 2|134, 3|124, 4|123$, and ii) $(n_A = 2, n_B = 2)$ that corresponds to the $N_{n_B} = 3$ bipartitions, $\vec{\alpha}|\vec{\beta} = 12|34, 13|24, 14|23$. The total number of bipartitions in this case is $L = N_{n_A=1} + N_{n_A=2} = 4 + 3 = 2^{4-1} - 1 = 7$. The table II presents some examples with $n = 1, \dots, 6$ for the number of classes (n_A^{max}) and partitions (L).

It was a great advance in the understanding of multipartite entanglement to distinguish full n -partite inseparable states (definition below) from those states that have genuine multipartite entanglement [35–37]. A genuine n -partite entangled state is one that does not belong to the

family of “biseparable” states [35, 36]:

$$\begin{aligned}\hat{\rho}_{bs} &\equiv \sum_{\{\vec{\alpha}|\vec{\beta}\}} p_{\{\vec{\alpha}|\vec{\beta}\}} \hat{\rho}_{\{\vec{\alpha}|\vec{\beta}\}} = \\ &= \sum_{\{\vec{\alpha}|\vec{\beta}\}} p_{\{\vec{\alpha}|\vec{\beta}\}} \left(\sum_j \eta_j^{\{\vec{\alpha}|\vec{\beta}\}} \hat{\rho}_j^{\{\vec{\alpha}\}} \otimes \hat{\rho}_j^{\{\vec{\beta}\}} \right), \quad (13)\end{aligned}$$

where the first sum in Eq.(13) runs over the set $\{\vec{\alpha}|\vec{\beta}\}$ of all the L possible bipartitions of the system and the states in between parenthesis are generic separable states in the bipartition $\vec{\alpha}|\vec{\beta}$. Normalization requires that $\sum_{\{\vec{\alpha}|\vec{\beta}\}} p_{\{\vec{\alpha}|\vec{\beta}\}} = 1 = \sum_j \eta_j^{\{\vec{\alpha}|\vec{\beta}\}}$. Note that for a given class, all the modes in a given part might be entangled. Thus, all forms of genuine n_A - or n_B -partite entanglement may appear in state (13).

Any genuine multipartite entanglement criterion needs to test entanglement in all the possible bipartitions that can be drawn in the system. However, testing bipartitions individually is not enough, since biseparable states (13) could be entangled in every bipartition of the system. These are called full n -partite inseparable states [36, 37]. For example, a class of n -partite inseparable states corresponds to biseparable states with all the coefficients $p_{\{\vec{\alpha}|\vec{\beta}\}}$ different from zero in Eq.(13), although this is not a necessary condition (see a 3-mode example in [37]). Therefore, a genuine entanglement criteria needs to refute biseparable state $\hat{\rho}_{bs}$ as a possible description of the system.

In the following section we first introduce entanglement criteria to test entanglement in any bipartition of an n -mode system, and then use these criteria to construct criteria for genuine n -partite entanglement.

IV. ENTANGLEMENT CRITERIA FOR GENERIC BIPARTITIONS

Here we consider that the multipartite system consists of n bosonic modes, described through local canonical operators:

$$\hat{\mathbf{z}} \equiv (\hat{\mathbf{x}}, \hat{\mathbf{p}})^T = (\hat{x}_1, \dots, \hat{x}_n, \hat{p}_1, \dots, \hat{p}_n)^T, \quad (14)$$

where T means transposition. Each pair of canonically conjugate observables with continuous spectra x_j and p_j verifies the commutation relation $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$ and we generically call them position and momentum, respectively. Here we'll prove that the general form introduced in Eq.(7) for bipartite entanglement criteria can be applied not only for a two mode system but also for the general case of an arbitrary bipartition $\vec{\alpha}|\vec{\beta}$ of an n -mode bosonic system. We indicate the partial transposition as $T_{\vec{t}}$, with $\vec{t} = \vec{\alpha}$ if the partial transposition is with respect to the modes contained in $\vec{\alpha}$ and $\vec{t} = \vec{\beta}$ if the partial transposition is with respect to the modes contained in $\vec{\beta}$. Because each of the $L = 2^n - 1$ possible bipartitions

can be tested with a different pair of non-local operators, we rewrite Eq.(7) as

$$F[\hat{\rho}^{T_{\vec{t}}}, P_{\hat{u}_{\vec{t}}}, P_{\hat{v}_{\vec{t}}}] = F[\hat{\rho}, P_{\hat{\mu}_{\vec{t}}}, P_{\hat{\nu}_{\vec{t}}}] \geq f(|\vec{\delta}_{\vec{t}}|), \quad (15)$$

where $|\vec{\delta}_{\vec{t}}| = \max\{|\gamma_{\vec{t}}|, |\delta_{\vec{t}}|\}$ with $[\hat{u}_{\vec{t}}, \hat{v}_{\vec{t}}] = i\gamma_{\vec{t}}\hat{\mathbb{1}}$ and $[\hat{\mu}_{\vec{t}}, \hat{\nu}_{\vec{t}}] = i\delta_{\vec{t}}\hat{\mathbb{1}}$. The objective here is to determine what entanglement criteria can be tested, given that the experimentalist has access to marginal probability distributions of observables (16). Therefore, in the following we will show that for a fixed pair of operators:

$$\hat{\mu}_{\vec{t}} \equiv \sum_{j=1}^n h_j \hat{x}_j, \quad (16a)$$

$$\hat{\nu}_{\vec{t}} \equiv \sum_{j=1}^n g_j \hat{p}_j, \quad (16b)$$

with commutator

$$\delta_{\vec{t}} = \sum_{j=1}^n h_j g_j, \quad (17)$$

where h_j, g_j are arbitrarily real numbers, it is always possible to find mirrored operators $\hat{u}_{\vec{t}}$ and $\hat{v}_{\vec{t}}$ that satisfy the equality in Eq. (15). But, first note that if we want to accommodate observables of the form $\hat{\mu}_{\vec{t}} = \sum_{j=1}^n h'_j \hat{x}'_j + g'_j \hat{p}'_j$ and $\hat{\nu}_{\vec{t}} = \sum_{j=1}^n h''_j \hat{x}'_j + g''_j \hat{p}'_j$, we simply redefine the quadrature variables as $h_j \hat{x}_j = h'_j \hat{x}'_j + g'_j \hat{p}'_j$ and $g_j \hat{p}_j = h''_j \hat{x}'_j + g''_j \hat{p}'_j$.

All of the possible non-local operators of the n -mode system are defined as linear combinations of the local operators written in (14). We can write them as

$$\hat{\mathbf{z}}_{\vec{t}} \equiv (\hat{\mathbf{u}}_{\vec{t}}, \hat{\mathbf{v}}_{\vec{t}})^T = \mathbb{M}_{\vec{t}} \hat{\mathbf{z}}, \quad (18)$$

where

$$(\hat{\mathbf{u}}_{\vec{t}}, \hat{\mathbf{v}}_{\vec{t}})^T \equiv (\hat{u}_{1,\vec{t}}, \dots, \hat{u}_{n,\vec{t}}, \hat{v}_{1,\vec{t}}, \dots, \hat{v}_{n,\vec{t}})^T, \quad (19)$$

is the $2n$ -component vector of possible linear combinations. The matrix $\mathbb{M}_{\vec{t}} \equiv \text{diag}(\mathbb{M}_{x,\vec{t}}, \mathbb{M}_{p,\vec{t}})$ is a $2n \times 2n$ real matrix and $\mathbb{M}_{x,\vec{t}}$ and $\mathbb{M}_{p,\vec{t}}$ are non-singular real $n \times n$ matrices that we call the x -matrix and the p -matrix of the bipartition, respectively. First, we identify the mirrored observables in Eq.(15) as

$$\hat{u}_{\vec{t}} \equiv \hat{u}_{k,\vec{t}} = \sum_{j=1}^n (\mathbb{M}_{x,\vec{t}})_{k,j} \hat{x}_j, \quad (20a)$$

$$\hat{v}_{\vec{t}} \equiv \hat{v}_{k,\vec{t}} = \sum_{j=1}^n (\mathbb{M}_{p,\vec{t}})_{k,j} \hat{p}_j. \quad (20b)$$

What remains is to determine the rows of the matrix $\mathbb{M}_{\vec{t}}$ such that the equality $F[\hat{\rho}^{T_{\vec{t}}}, P_{\hat{u}_{\vec{t}}}, P_{\hat{v}_{\vec{t}}}] = F[\hat{\rho}, P_{\hat{\mu}_{\vec{t}}}, P_{\hat{\nu}_{\vec{t}}}]$ in Eq.(15) holds for any functional and for the measurable operators (16).

Since the local operators satisfy $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$, we impose that the non-local operators satisfy the commutation relation $[\hat{u}_{j,\vec{t}}, \hat{v}_{k,\vec{t}}] = i\gamma_{\vec{t}}\delta_{jk}$ with $\gamma_{\vec{t}}$ any real number.

This means that: $\mathbb{M}_{p,\vec{t}}^T \mathbb{M}_{x,\vec{t}} = \mathbb{M}_{x,\vec{t}} \mathbb{M}_{p,\vec{t}}^T = \gamma_{\vec{t}} \mathbb{1}$ [50], so the x-matrix is determined by the p-matrix:

$$\mathbb{M}_{x,\vec{t}} = \gamma_{\vec{t}} (\mathbb{M}_{p,\vec{t}}^{-1})^T. \quad (21)$$

When the matrix $\mathbb{M}_{\vec{t}}$ is orthogonal, *i.e.* $\mathbb{M}_{\vec{t}}^T = \mathbb{M}_{\vec{t}}^{-1}$, we have that each pair of non-local observables $\hat{u}_{j,\vec{t}}$ and $\hat{v}_{j,\vec{t}}$ ($j = 1, \dots, n$) are conjugate operators. In this particular case Eq.(21) reduce to:

$$\mathbb{M}_{x,\vec{t}} = \gamma_{\vec{t}} (\mathbb{M}_{p,\vec{t}}). \quad (22)$$

In Appendixes A and B, we prove that the coefficients in Eqs. (20) are uniquely given by the k^{th} row:

$$(\mathbb{M}_{p,\vec{t}})_{kj} = \bar{g}_j = \begin{cases} -g_j & \text{if } j \text{ is one component of } \vec{t} \\ g_j & \text{otherwise} \end{cases} \quad (23)$$

and

$$((\mathbb{M}_{p,\vec{t}}^{-1})^T)_{k,j} = \frac{h_j}{\gamma_{\vec{t}}}. \quad (24)$$

Therefore, the commutator between the mirrored observables is $[\hat{u}_{\vec{t}}, \hat{v}_{\vec{t}}] = i\gamma_{\vec{t}} \mathbb{1}$, with

$$\gamma_{\vec{t}} = \sum_{j=1}^n h_j \bar{g}_j = \pm \gamma_{\vec{\alpha}}, \quad (25)$$

where the plus sign is when $\vec{t} = \vec{\alpha}$ and the minus sign is when $\vec{t} = \vec{\beta}$.

In this way, we see that the coefficients of the mirrored operators $\hat{u}_{\vec{t}}$ and $\hat{v}_{\vec{t}}$ are determined once we specify the p-matrix of the bipartition $\mathbb{M}_{p,\vec{t}}$, whose k^{th} row is given in Eq.(23) and with the k^{th} row of $(\mathbb{M}_{p,\vec{t}}^{-1})^T$ given in Eq.(24). Without loss of generality we can choose $k = 1$. In Appendix B we give the general structure of a matrix $\mathbb{M}_{p,\vec{t}}$ with these properties that satisfied (21). This proves the equality $F[\hat{\rho}^{T_{\vec{t}}}, P_{\hat{u}_{\vec{t}}}, P_{\hat{v}_{\vec{t}}}] = F[\hat{\rho}, P_{\hat{u}_{\vec{t}}}, P_{\hat{v}_{\vec{t}}}]$ in Eq.(15).

For states $\hat{\rho}_{\{\vec{\alpha}|\vec{\beta}\}}$ that are separable in the bipartition $\vec{\alpha}|\vec{\beta}$, the partial transposed state $\hat{\rho}_{\{\vec{\alpha}|\vec{\beta}\}}^{T_{\vec{t}}}$ (with $\vec{t} = \vec{\alpha}$ or $\vec{t} = \vec{\beta}$) is also a physical state. In this case, Eq.(15) is not violated, since it is a valid uncertainty relation for both sets of operators. Therefore, violation of Eq. (15) constitutes a bipartite entanglement criterion for n modes in bipartition $\vec{\alpha}|\vec{\beta}$, where the lower bound is given by

$$|\bar{\delta}_{\vec{t}}| = \max\{|\gamma_{\vec{t}}|, |\delta_{\vec{t}}|\} = \sum_{j \notin \{\vec{t}\}} |h_j g_j| + \sum_{j \in \{\vec{t}\}} |h_j \bar{g}_j|. \quad (26)$$

Here the first sum runs over indexes j corresponding to those modes that were not transposed and the second over those modes that were transposed. To observe the violation of inequality (15) for states entangled in the bipartition it is necessary to choose the coefficients of the operators (h_j and g_j) in a way such that $|\bar{\delta}_{\vec{t}}| = |\gamma_{\vec{t}}| > |\delta_{\vec{t}}|$.

When we apply the sum of variance functional $F = F_{Lin}$ defined in Eq.(8) with $f(|\delta_{\vec{t}}|) = |\bar{\delta}_{\vec{t}}|$ in Eq.(15) we recover the van Loock and Furusawa entanglement criterion in Eq. (28) of Ref. [31]. The factor of 1/2 of the lower bound in Ref. [31] is due to the fact that there the commutation relation is $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}/2$ instead of $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$, as defined here. We stress that although the PPT argument is not used explicitly in Ref. [31], here we have shown these sum of variance inequalities indeed belong to the class of PPT+URNLO bipartite entanglement criteria.

We stress that we have proved that for every pair of linear non-local observables $(\hat{u}_{\vec{t}}, \hat{v}_{\vec{t}})$, it is always possible to find a pair of mirrored linear non-local observables $(\hat{u}_{\vec{t}}, \hat{v}_{\vec{t}})$ such that the equality in Eq. (15) holds. This relies on the structure of the matrix $\mathbb{M}_{p,\vec{t}}$ in Appendix B. In particular, we can always choose to test entanglement with any pair of commuting non-local observables (*i.e.* $[\hat{u}_{\vec{t}}, \hat{v}_{\vec{t}}] = \delta_{\vec{t}} = 0$), with the advantage that in this case we simply have a lower bound given by $|\bar{\delta}_{\vec{t}}| = |\gamma_{\vec{\alpha}}|$. Moreover, the inequality can always be violated by a simultaneous eigenstate of the commuting operators, which are entangled states, and are the backbone of quantum information in CV systems [45, 46]. Examples of these type of eigenstates are the 2-mode EPR states that are simultaneous eigenstates of relative position $\hat{u} = \hat{x}_1 - \hat{x}_2$ and the total momentum $\hat{v} = \hat{p}_1 + \hat{p}_2$ [47], or the CV GHZ n -modes states that are simultaneous eigenstates of the relative positions $\hat{u}_1 = \hat{x}_1 - \hat{x}_2, \hat{u}_2 = \hat{x}_2 - \hat{x}_3, \dots, \hat{u}_{n-1} = \hat{x}_{n-1} - \hat{x}_n$ and the total momentum $\hat{v} = \hat{p}_1 + \dots + \hat{p}_n$ [31]. It is worth noting that these type of unnormalised eigenstates are well approximated by squeezed multipartite gaussian states that have been generated in several experiments recently [12–14]. In the next section we use this type of commuting non-local linear operators to derive criteria that are useful to detect genuine multipartite entanglement in any n -partite state.

V. GENUINE MULTIPARTITE PPT+URNLO ENTANGLEMENT CRITERIA

Here we will derive genuine PPT+URNLO entanglement criteria that exclude the possibility of a violation by the set of biseparable states $\hat{\rho}_{bs}$ given in Eq.(13). We will provide two different types of genuine entanglement criteria: a first one based on a single pair of commuting operators and a second one based on a set of pairs of commuting operators.

A. Criteria with a single pair of commuting non-local operators

Let us suppose that we can find a pair of suitable non-local operators $\hat{\mu} = \sum_{j=1}^n h_j \hat{x}_j$ and $\hat{\nu} = \sum_{j=1}^n g_j \hat{p}_j$ such that $[\hat{\mu}, \hat{\nu}] = 0$ with $\gamma_{\vec{\alpha}} = \sum_{j=1}^n h_j \bar{g}_j \neq 0$ for all the L

bipartition's of the system (where $\bar{g}_j = -g_j$ if j is one component of the vector $\vec{\alpha}$ or $\bar{g}_j = g_j$ otherwise). This means we must consider all possible different location of minus sign in \bar{g}_j within $\gamma_{\vec{\alpha}}$ defined in Eq.(25), for fixed values of the coefficients h_j and g_j . Then in this case we could test entanglement in all the bipartition of the system through the different inequalities (see Eq.(15)),

$$F[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] \geq f(|\gamma_{\vec{\alpha}}|), \quad (27)$$

but with a single pair $(\hat{\mu}, \hat{\nu})$ of non-local operators. Before we prove that there is always a family of such pairs in an n -mode system, let us see that if $\gamma_{\min} = \min_{\{\vec{\alpha}\}} \{|\gamma_{\vec{\alpha}}|\} \geq 0$, and $\{\vec{\alpha}\}$ runs over all the L bipartitions of the system, then the single inequality,

$$F[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] \geq f(\gamma_{\min}) \geq 0, \quad (28)$$

is never violated by biseparable states defined in Eq. (13). So, violation of this single inequality constitutes a genuine multipartite entanglement criterion because it tests entanglement in all the bipartitions of the system and it is never violated by biseparable states. In order to prove this, one of two conditions must be true: i) the functional F is concave with respect to a convex sum of density operators, *i.e.* $F[\hat{\rho} = \sum_j p_j \hat{\rho}_j, P_{\hat{\mu}}, P_{\hat{\nu}}] \geq \sum_j p_j F[\hat{\rho}_j, P_{\hat{\mu}}, P_{\hat{\nu}}]$ or ii) the functional $F[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] \geq F'[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}]$ where F' is concave with respect to a convex sum of density operators. If the first condition is valid, then for biseparable states $\hat{\rho}_{bs}$ we have,

$$\begin{aligned} F[\hat{\rho}_{bs}, P_{\hat{\mu}}, P_{\hat{\nu}}] &\geq \sum_{\{\vec{\alpha}|\vec{\beta}\}} p_{\{\vec{\alpha}|\vec{\beta}\}} F[\hat{\rho}_{\{\vec{\alpha}|\vec{\beta}\}}, P_{\hat{\mu}}, P_{\hat{\nu}}] \\ &\geq \sum_{\{\vec{\alpha}|\vec{\beta}\}} p_{\{\vec{\alpha}|\vec{\beta}\}} f(|\gamma_{\vec{\alpha}}|) \\ &\geq f(\gamma_{\min}) \geq 0, \end{aligned} \quad (29)$$

where we assume by hypothesis that the inequality in Eq.(27) is never violated by separable states $\hat{\rho}_{\{\vec{\alpha}|\vec{\beta}\}}$ in the bipartition $\vec{\alpha}|\vec{\beta}$. Therefore, it can be seen immediately that the same is true if the second condition is valid. In Appendix C we prove the concavity of the sum of entropies functional F_E . This result, together with the chain of inequalities in Eq.(11), proves that, besides the entropic UR, we can use either the sum of variances UR in Eq.(8) or the product of variances UR in Eq.(9) to set the functional F in our genuine entanglement criterion in Eq.(28).

In what follows we develop a systematic way to find the non-local commuting operators in the genuine entanglement criterion in Eq.(28). Let's start by defining $\mathbb{A}_{\vec{t}}$ as the diagonal matrix with ones in the location of modes that are not transposed and negative ones in the location of modes that are transposed. We consider a "seed" partition, defined by the vector $\vec{s} \equiv (1, \dots, n_A)$ of modes in A . From this seed partition corresponding to a given class (n_A, n_B) , we can generate all elements of this class

by swapping the modes around. To do this we define the bipartition permutation matrix $\mathbb{P}_{\vec{s}\vec{\alpha}} = \prod_{i=1}^{n_A} \mathbb{P}_{i\alpha_i}$, where $\mathbb{P}_{i\alpha_i}$ is the permutation matrix between mode i and mode α_i . In other words, $\mathbb{P}_{i\alpha_i}$ is the matrix obtained from swapping the rows i and α_i of the identity matrix $\mathbb{1}$, and $\mathbb{P}_{i,i} = \mathbb{1}$. Note that $\mathbb{P}_{\vec{s}\vec{\alpha}} \mathbb{P}_{\vec{s}\vec{\alpha}}^T = \mathbb{1}$ because $\mathbb{P}_{i\alpha_i}^2 = \mathbb{1}$. Therefore, for an arbitrary matrix \mathbb{M} , the matrix $\mathbb{P}_{\vec{s}\vec{\alpha}} \mathbb{M}$ corresponds to swapping all the rows in \mathbb{M} indicated by \vec{s} with all the target rows indicated by $\vec{\alpha}$. Analogously, the matrix $\mathbb{M} \mathbb{P}_{\vec{s}\vec{\alpha}}^T$ corresponds to swapping all the columns in \mathbb{M} indicated by \vec{s} with all the target columns indicated by $\vec{\alpha}$.

First, we observe that we can write the matrix $\mathbb{A}_{\vec{t}}$, associated with the partial transposition with respect to any set of modes indicated in the vector \vec{t} , as:

$$\mathbb{A}_{\vec{t}} = \pm \mathbb{P}_{\vec{s}\vec{\alpha}} \mathbb{A}_{\vec{s}} \mathbb{P}_{\vec{s}\vec{\alpha}}^T, \quad (30)$$

where the minus sign applies when $\vec{t} = \vec{\beta}$ and the plus sign when $\vec{t} = \vec{\alpha}$.

We call $\mathbb{A}_{\vec{s}}$, $\mathbb{M}_{p,\vec{s}}$ and $\mathbb{M}_{x,\vec{s}} = \gamma_{\vec{s}} (\mathbb{M}_{p,\vec{s}}^{-1})^T$ the "seed" matrices associated with the bipartition class (n_A, n_B) . With these matrices we can express the operators defined in Eqs. (16) for the seed bipartition as

$$\hat{\mu}_{1,\vec{s}} = \sum_{j=1}^n (\mathbb{M}_{x,\vec{s}})_{1j} \hat{x}_j, \quad \hat{\nu}_{1,\vec{s}} = \sum_{j=1}^n (\mathbb{M}_{p,\vec{s}} \mathbb{A}_{\vec{s}})_{1j} \hat{p}_j. \quad (31)$$

Furthermore, we impose that $[\hat{\mu}_{1,\vec{s}}, \hat{\nu}_{1,\vec{s}}] = i(\mathbb{M}_{p,\vec{s}} \mathbb{A}_{\vec{s}} \mathbb{M}_{x,\vec{s}}^T)_{11} = i\delta_{\vec{s}} = 0$. Therefore, according to Eqs.(20), the mirrored non-local operators are:

$$\hat{u}_{1,\vec{s}} = \hat{\mu}_{1,\vec{s}} \quad \text{and} \quad \hat{v}_{1,\vec{s}} = \sum_{j=1}^n (\mathbb{M}_{p,\vec{s}})_{1j} \hat{p}_j, \quad (32)$$

such that $[\hat{u}_{1,\vec{s}}, \hat{v}_{1,\vec{s}}] = i(\mathbb{M}_{p,\vec{s}} \mathbb{M}_{x,\vec{s}}^T)_{11} = i\gamma_{\vec{s}} \neq 0$. We can rephrase these statements by saying that given the coefficient of the mirrored operators $\hat{u}_{\vec{s}}$ and $\hat{v}_{\vec{s}}$ corresponding to the first columns of the seed matrices $(\mathbb{M}_{x,\vec{s}})_{1j} = (h_1, \dots, h_{n_A}, h_{n_A+1}, \dots, h_n)$ and $(\mathbb{M}_{p,\vec{s}})_{1j} = (-g_1, \dots, -g_{n_A}, g_{n_A+1}, \dots, g_n)$, respectively, the coefficient of the operators $\hat{\mu}_{\vec{s}}$ and $\hat{\nu}_{\vec{s}}$ to test entanglement in the seed bipartition $\vec{s}|\vec{\beta}$ are $(\mathbb{M}_{x,\vec{s}})_{1j} = (h_1, \dots, h_{n_A}, h_{n_A+1}, \dots, h_n)$ and $(\mathbb{M}_{p,\vec{s}} \mathbb{A}_{\vec{s}})_{1j} = (g_1, \dots, g_{n_A}, g_{n_A+1}, \dots, g_n)$, respectively. Of course the condition that $\hat{\mu}_{\vec{s}}$ and $\hat{\nu}_{\vec{s}}$ commute impose some restrictions on the possible coefficients h_j and g_j .

In order to test entanglement in the rest of the bipartitions of the same class (n_A, n_B) , we can use the matrices

$$\mathbb{M}_{p,\vec{t}} = \mathbb{P}_{\vec{s}\vec{\alpha}} \mathbb{M}_{p,\vec{s}} \mathbb{P}_{\vec{s}\vec{\alpha}}^T, \quad \mathbb{M}_{x,\vec{t}} = \gamma_{\vec{s}} \mathbb{P}_{\vec{s}\vec{\alpha}} (\mathbb{M}_{p,\vec{s}}^{-1})^T \mathbb{P}_{\vec{s}\vec{\alpha}}^T. \quad (33)$$

One can check that these also verify Eq. (21). This means that the first rows of the seed matrices $\mathbb{M}_{x,\vec{s}}$ and $\mathbb{M}_{p,\vec{s}}$ were scrambled by the matrix $\mathbb{P}_{\vec{s}\vec{\alpha}}^T$ and then mapped to the α_1 row by the matrix $\mathbb{P}_{\vec{s}\vec{\alpha}}$. Thus, from (31) the

non-local operators to test bipartition $\vec{\alpha}|\vec{\beta}$ are:

$$\hat{\mu}_{\alpha_1, \vec{t}} = \sum_{j=1}^n (\mathbb{M}_{x, \vec{s}} \mathbb{P}_{\vec{s}\vec{\alpha}}^T)_{\alpha_1, j} \hat{x}_j \quad (34a)$$

$$\hat{\nu}_{\alpha_1, \vec{t}} = \pm \sum_{j=1}^n (\mathbb{M}_{p, \vec{s}} \mathbb{A}_{\vec{s}} \mathbb{P}_{\vec{s}\vec{\alpha}}^T)_{\alpha_1, j} \hat{x}_j. \quad (34b)$$

We emphasize that although the mirrored operators

$$\hat{u}_{\alpha_1, \vec{t}} = \hat{\mu}_{\alpha_1, \vec{t}} \text{ and } \hat{v}_{\alpha_1, \vec{t}} = \sum_{l=1}^n (\mathbb{M}_{p, \vec{s}} \mathbb{P}_{\vec{s}\vec{\alpha}}^T)_{\alpha_1, l} \hat{x}_l, \quad (35)$$

are different from the mirrored operators in Eq.(32) for the seed bipartition, their commutator $[\hat{u}_{\alpha_1, \vec{t}}, \hat{v}_{\alpha_1, \vec{t}}] = [\hat{u}_{1, \vec{s}}, \hat{v}_{1, \vec{s}}] = i\gamma_{\vec{s}}$ is the same. So, using the matrices in Eqs.(33), the lower bound in Eq.(27) is equal for all bipartitions of the same class (n_A, n_B) . Thus, we can use this result to obtain a single pair of non-local operators to be used in the genuine multipartite entanglement criterion in Eq.(28).

For an arbitrary bipartition class $(n_A, n - n_A)$ ($1 \leq n_A \leq n_A^{\max}$) we set two type of seed matrices: $\tilde{\mathbb{M}}_{p, \vec{s}}$ and $\tilde{\mathbb{M}}'_{p, \vec{s}}$ respectively. For bipartitions $\vec{\alpha}|\vec{\beta}$ such that $n \notin \vec{\alpha}$, we choose the seed matrix $\tilde{\mathbb{M}}_{p, \vec{s}}$ with the structure given in Eq.(B4) of Appendix B. It's first row is:

$$(\tilde{\mathbb{M}}_{p, \vec{s}})_{1j} = (-g, \dots, -g, g, \dots, g, g'), \quad (36)$$

where the minus sign appears in the first n_A positions, $g = -\gamma/2h$ and $g' = \gamma(n-1)/2h'$ (h, h', γ arbitrary real numbers). For bipartitions such that $n \in \vec{\alpha}$, we choose the seed matrix $\tilde{\mathbb{M}}'_{p, \vec{s}}$ whose first row is:

$$(\tilde{\mathbb{M}}'_{p, \vec{s}})_{1j} = (-g, \dots, -g', \dots, -g, \dots, -g, g, \dots, g), \quad (37)$$

where again the minus sign stands in the first n_A positions and g' is located in the position i ($1 \leq i \leq n_A$) such that $\alpha_i = n$ with α_i a component of $\vec{\alpha}$. Therefore, because it is always true that $\mathbb{M}_{x, \vec{s}} = \gamma_{\vec{s}} (\mathbb{M}_{p, \vec{s}}^{-1})^T$, for the x-matrix associated with the p-matrix in Eq.(36) (*i.e.* when $n \notin \vec{\alpha}$) we have,

$$(\tilde{\mathbb{M}}_{x, \vec{s}})_{1j} = (h, h, \dots, h, h'). \quad (38)$$

And for the x-matrix associated with the p-matrix in Eq.(37), (*i.e.* when $n \in \vec{\alpha}$) we have,

$$(\tilde{\mathbb{M}}'_{x, \vec{s}})_{1j} = (h, \dots, h', h, \dots, h), \quad (39)$$

where the location of h' is in the position i ($1 \leq i \leq n_A$) such that $\alpha_i = n$. These seed matrices set the mirrored operators in Eqs.(32) and (35) with the commutator,

$$[\hat{u}_{l_1, \vec{t}}, \hat{v}_{l_1, \vec{t}}] = [\hat{u}_{1, \vec{s}}, \hat{v}_{1, \vec{s}}] = in_A \gamma, \quad (40)$$

if the mode $n \notin \vec{\alpha}$ and

$$[\hat{u}_{l_1, \vec{s}}, \hat{v}_{l_1, \vec{s}}] = [\hat{u}_{1, \vec{s}}, \hat{v}_{1, \vec{s}}] = -i(n - n_A) \gamma, \quad (41)$$

if the mode $n \in \vec{\alpha}$. Then, because $(\tilde{\mathbb{M}}'_{p, \vec{s}} \mathbb{A}_{\vec{s}} \mathbb{P}_{\vec{s}\vec{\alpha}}^T)_{\alpha_1 j} = (\tilde{\mathbb{M}}_{p, \vec{s}} \mathbb{A}_{\vec{s}} \mathbb{P}_{\vec{s}\vec{\alpha}}^T)_{\alpha_1 j} = (g, \dots, g, g')$ and $(\tilde{\mathbb{M}}'_{x, \vec{s}} \mathbb{P}_{\vec{s}\vec{\alpha}}^T)_{\alpha_1 j} = (\tilde{\mathbb{M}}_{x, \vec{s}} \mathbb{P}_{\vec{s}\vec{\alpha}}^T)_{\alpha_1 j} = (h, \dots, h, h')$, the commuting operators in Eqs.(31) and (34) to test entanglement in all the bipartition of the class $(n_A, n - n_A)$ are equal to:

$$\hat{\mu} = h\hat{x}_1 + h\hat{x}_2 + \dots + h\hat{x}_{n-1} + h'\hat{x}_n \quad (42a)$$

$$\hat{\nu} = \frac{\gamma}{2} \left(-\frac{\hat{p}_1}{h} - \dots - \frac{\hat{p}_{n-1}}{h} + \frac{(n-1)\hat{p}_n}{h'} \right), \quad (42b)$$

with $[\hat{\mu}, \hat{\nu}] = 0$. This family of single pairs of operators are the ones that must be used in our genuine entanglement criterion in Eq.(28) with $\gamma_{\min} = \min_{\{\vec{\alpha}\}} \{|\gamma_{\vec{\alpha}}|\} = \min_{n_A} \{n_A |\gamma|, (n - n_A) |\gamma|\} = |\gamma|$, where the minimization is over the values $1 \leq n_A \leq n_A^{\max}$. The three free parameters of this family (h, h', γ) increases the chances to detect genuine entanglement in a n -mode systems using the single inequality in Eq.(28).

We recover the commuting non-local operators that appear in Eq.(30) of [31], if we relabel mode 1 as n and set $h' = 1$ and $h = -1/\sqrt{n-1}$, then $\gamma = 2/(n-1)$. Here the lower bound must be given by $\gamma = 2/(n-1)$ according to our Eq.(28) with $F = F_{Lin}$ and $f_{Lin}(|\gamma|) = |\gamma| = 2/(n-1)$. The factor of 1/2 of the lower bound in Ref. [31] is due to the fact that there the commutation relation is $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}/2$ instead of $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$, as defined here. This proves that the popular entanglement criteria in [31] is indeed a PPT criteria. Although, in [31] it was not proven that the single inequality in our Eq.(28) with $F = F_{Lin}$ and $f_{Lin}(|\gamma|) = |\gamma| = 2/(n-1)$ is never violated by biseparable states $\hat{\rho}_{bs}$, this was pointed out in [36]. However, here we have gone a step further by proving that not only the sum of variance UR, but in fact any UR of the form Eq.(2) can be used to set a genuine entanglement criteria with (28) for any pair of linear non-local operators of the family (42). In this regard, we stress that with similar experimental effort one can test entropic entanglement criteria, which outperform variance criteria in general. It is worth noting that the genuine tripartite entanglement criteria in Eq.(1) of [38] is a special case of our genuine criterion given in (28) with $F = F_H$ and $f_H(|\gamma|) = |\gamma|^2/4$ where the single pair of operators used $\mu = \hat{x}_1 - (x_2 + x_3)/\sqrt{2}$ and $\nu = p_1 + (p_2 + p_3)/\sqrt{2}$ can be mapped in our family of commuting operators in (42) if we rename mode 1 as mode 3 and set $h = -1/\sqrt{2}$ and $h' = \gamma = 1$. The lower bound in Ref. [38] is 1 instead of $f_H(|\gamma| = 1) = 1/4$ of our case due to the fact that there the commutation relation is $[\hat{x}_j, \hat{p}_k] = 2i\delta_{jk}$ instead of $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$, as defined here.

A limitation of using the single inequality in Eq.(28) as a genuine entanglement criterion is that the range of URs that can be used to set the functional F (and the function f) is restricted to those URs of the form in Eq.(2) valid for arbitrary linear non-local observables, excluding the possibility to use URs valid for conjugate pairs, *i.e.* for those pairs of operators related by a $\pi/2$ rotation. For example, we can not use the UR in Eq.(12) to set the functional F and the function f in our PPT-

URNLO genuine entanglement criterion in Eq.(28). This is because the pairs of mirrored operators $(\hat{u}_{l_1, \vec{s}}, \hat{v}_{l_1, \vec{t}})$ and $(\hat{u}_{1, \vec{s}}, \hat{v}_{1, \vec{s}})$ associated with the pairs of operators $\hat{\mu}$ and $\hat{\nu}$ in Eq.(42) cannot be canonical conjugated. In order to see this, notice that the seed matrices $\tilde{M}_{p, \vec{s}}$ and $\tilde{M}'_{p, \vec{s}}$, whose first rows are in Eqs.(36) and (37) respectively, and the matrices $\tilde{M}_{x, \vec{s}}$ and $\tilde{M}'_{x, \vec{s}}$, whose first rows are in Eq.(38) and (39) respectively, do not satisfy the condition in Eq.(22), that guarantee the conjugation between the mirrored operators, for any value of the parameters h, h', γ . We can fix this limitation if we consider PPT-URNLO entanglement criteria based on a set of pairs of commuting operators.

B. Criterion with several pairs of commuting non-local operators

It is also possible to have a genuine PPT-URNLO entanglement criterion that consists also in a single inequality but, contrary to the one in Eq.(28), involves several pairs of commuting operators. We start defining a set of pairs $\{(\hat{\mu}_m, \hat{\nu}_m)\}$ of commuting operators with $m = 1, \dots, M$ ($M \leq L$) of the form $\hat{\mu}_m = \sum_{j=1}^n h_{mj} \hat{x}_j$ and $\hat{\nu}_m = \sum_{j=1}^n g_{mj} \hat{p}_j$, where h_{mj}, g_{mj} are real numbers not all equal to zero. We also define $\gamma_{m, \vec{\alpha}} \equiv \sum_{j=1}^n \bar{g}_{mj} h_{mj}$ where for all values of m , $\bar{g}_{mj} = -g_{mj}$ if j is one component of the vector $\vec{\alpha}$ or $\bar{g}_{mj} = g_{mj}$ otherwise. Now, for a given set of coefficients h_{mj} and g_{mj} (m fixed and $j = 1, \dots, n$), we must check all the possible locations of the minus sign of \bar{g}_{mj} within $\gamma_{m, \vec{\alpha}} \equiv \sum_{j=1}^n \bar{g}_{mj} h_{mj}$ and see for which combinations of locations $\gamma_{m, \vec{\alpha}}$ is different from zero. According to Eq.(25), for m fixed, the number $\gamma_{m, \vec{\alpha}}$ is the commutator of the operator pair (\hat{u}_m, \hat{v}_m) where the location of the minus sign in $\gamma_{m, \vec{\alpha}}$ indicates which modes were partial transposed.

We denote the subset of all bipartitions $\{\vec{\alpha}|\vec{\beta}\}$ of the system where the values of $\gamma_{m, \vec{\alpha}}$ are not zero as $\{\vec{\alpha}\}_m$ (for every fixed value of m). This means, that we can test entanglement with the pair $(\hat{\mu}_m, \hat{\nu}_m)$ in all the bipartitions $\{\vec{\alpha}\}_m$ with the single inequality (like the one in Eq.(28)),

$$F[\hat{\rho}, P_{\hat{\mu}_m}, P_{\hat{\nu}_m}] \geq f(\gamma_{m, \min}) \geq 0, \quad (43)$$

where we call $\gamma_{m, \min} = \min_{\{\vec{\alpha}\}_m} \{|\gamma_{m, \vec{\alpha}}|\} > 0$. Notice, that for every pair of observables $(\hat{\mu}_m, \hat{\nu}_m)$ correspond a set $\{(\hat{u}_{m, l}, \hat{v}_{m, l})\}$ of mirrored observables, with $l = 1, \dots, L_m$ and L_m the total number of bipartitions in the set $\{\vec{\alpha}\}_m$. This means that for every bipartition in the set $\{\vec{\alpha}\}_m$, we have a different pair of mirrored observables $\hat{u}_{m, l} = \sum_{j=1}^n h_{m, j} \hat{x}_j$ and $\hat{v}_{m, l} = \sum_{j=1}^n \bar{g}_{m, j} \hat{p}_j$ (the index l is counting the different location of minus sign such $\gamma_{m, \vec{\alpha}} \equiv \sum_{j=1}^n \bar{g}_{mj} h_{mj} \neq 0$).

Now, consider functionals F such either i) are concave with respect to a convex sum of density operators or ii) verify $F > F'$ with F' concave with re-

spect to a convex sum of density operators. Then, we can proceed like in Eq.(29) and recognise that inequality (43) is never violated by biseparable states $\hat{\rho}_{bs}$ with $p_{\{\vec{\alpha}|\vec{\beta}\}} = 0$ corresponding to the bipartitions such that $\{\vec{\alpha}|\vec{\beta}\} \neq \{\vec{\alpha}\}_m$ (see Eq.(13)). However, states $\hat{\rho}_{bs}$ with $p_{\{\vec{\alpha}|\vec{\beta}\}} = 0$ for $\{\vec{\alpha}|\vec{\beta}\} = \{\vec{\alpha}\}_m$ do not necessarily verify the inequality (43). Furthermore, inequality (43) only tests bipartite entanglement in the bipartitions of the set $\{\vec{\alpha}\}_m$ (m fixed) that does not necessarily correspond to all the bipartitions of the system. But, if all of the sets $\{\{\vec{\alpha}\}_m, m = 1, \dots, M\}$ cover all possible bipartitions of the system (with possible repetitions), then all biseparable states $\hat{\rho} = \hat{\rho}_{bs}$ satisfy:

$$\sum_{m=1}^M F[\hat{\rho}, P_{\hat{\mu}_m}(\xi), P_{\hat{\nu}_m}(\xi)] \geq \lfloor \Theta \rfloor f(\tilde{\gamma}_{\min}) \geq f(\tilde{\gamma}_{\min}), \quad (44)$$

with $\tilde{\gamma}_{\min} = \min_m \{\gamma_{m, \min}\}$ and $\lfloor \Theta \rfloor = \lfloor \sum_{m=1}^M \sum_{\{\vec{\alpha}\}_m} p_{\{\vec{\alpha}\}_m} \rfloor \geq 1$, where $\lfloor x \rfloor$ is the largest integer not greater than x . Therefore, Eq.(44) constitutes a PPT-URNLO entanglement criterion for genuine n -partite entanglement.

Consider for example, in the case of 4-mode states, the set of commuting operators considered in Eq.(39) of Ref. [31] (see also [36]):

$$\begin{aligned} \hat{\mu}_1 &= \hat{x}_1 - \hat{x}_2, & \hat{\nu}_1 &= \hat{p}_1 + \hat{p}_2 + g_{13}\hat{p}_3 + g_{14}\hat{p}_4, \\ \hat{\mu}_2 &= \hat{x}_2 - \hat{x}_3, & \hat{\nu}_2 &= g_{21}\hat{p}_1 + \hat{p}_2 + \hat{p}_3 + g_{24}\hat{p}_4, \\ \hat{\mu}_3 &= \hat{x}_1 - \hat{x}_3, & \hat{\nu}_3 &= \hat{p}_1 + g_{32}\hat{p}_2 + \hat{p}_3 + g_{34}\hat{p}_4, \\ \hat{\mu}_4 &= \hat{x}_3 - \hat{x}_4, & \hat{\nu}_4 &= g_{41}\hat{p}_1 + g_{42}\hat{p}_2 + \hat{p}_3 + \hat{p}_4, \\ \hat{\mu}_5 &= \hat{x}_2 - \hat{x}_4, & \hat{\nu}_5 &= g_{51}\hat{p}_1 + \hat{p}_2 + g_{53}\hat{p}_3 + \hat{p}_4, \\ \hat{\mu}_6 &= \hat{x}_1 - \hat{x}_4, & \hat{\nu}_6 &= \hat{p}_1 + g_{62}\hat{p}_2 + g_{63}\hat{p}_3 + \hat{p}_4. \end{aligned} \quad (45)$$

For the first pair we have that $\gamma_{1, \vec{\alpha}} \equiv \sum_{j=1}^4 \bar{g}_{1j} h_{1j} \neq 0$ (where $g_{11} = g_{22} = h_{11} = 1$, $h_{12} = -1$ and $h_{13} = h_{14} = 0$) only when we consider partial transposition in the bipartitions: $\{\vec{\alpha}\}_1 = \{1|234, 2|134, 13|24, 14|23\}$. Therefore, with the pair $(\hat{\mu}_1, \hat{\nu}_1)$ we can test bipartite entanglement in the bipartitions in the set $\{\vec{\alpha}\}_1$ using the inequality in Eq.(43) with $\gamma_{1, \min} = 2$, since $\gamma_{1, \vec{\alpha}} = 2$ for all the bipartitions in the set $\{\vec{\alpha}\}_1$. Equivalently we can test bipartite entanglement with the rest of the pairs of operators in the following bipartitions: $\{\vec{\alpha}\}_2 = \{2|134, 3|124, 12|34, 13|24\}$, $\{\vec{\alpha}\}_3 = \{1|234, 3|124, 12|34, 14|23\}$, $\{\vec{\alpha}\}_4 = \{3|124, 4|123, 13|24, 14|23\}$, $\{\vec{\alpha}\}_5 = \{2|134, 4|123, 12|34, 14|23\}$, $\{\vec{\alpha}\}_6 = \{1|234, 4|123, 12|34, 13|24\}$ (where $\gamma_{m, \vec{\alpha}} = 2$ so $\tilde{\gamma}_{\min} = \gamma_{m, \min} = 2$ with $m = 1, \dots, 6$). Also, $\lfloor \Theta \rfloor = \lfloor 3(p_{1|234} + p_{2|134} + p_{3|124} + p_{4|123}) + 4(p_{12|34} + p_{13|24} + p_{14|23}) \rfloor = 3$ so we can use the lower bound $3f(\tilde{\gamma}_{\min})$ in our genuine entanglement criterion in Eq.(44). In the case when the functional is $F = F_{Lin}$ (and therefore $f(\tilde{\gamma}_{\min}) = f_{Lin}(\tilde{\gamma}_{\min}) = |\tilde{\gamma}_{\min}| = 2$) we recover the genuine entanglement criterion given in Eq.(43) of [36]. Again, the difference in the lower bound comes from the difference in the canonical commutation relation that they

used. We stress that the set of inequalities in Eq.(39) of Ref. [31], that correspond to our inequalities in Eq.(43) with the operators in Eq.(45) and $F = F_{Lin}$ ($f_{Lin}(\gamma_{m,\min}) = |\gamma_{m,\min}| = 2$), do not constitute a genuine multipartite entanglement criterion, as was pointed out in [36]. Only when the single inequality in Eq.(44) is used, a genuine multipartite entanglement criterion is achieved.

Also, we can easily recover the four-partite genuine entanglement criterion given in Eq.(44) of [36] if we use the set of pairs of non-local operators $\{(\hat{\mu}_m, \hat{\nu}_m)\}$ ($m = 1, 2$) where the pair $(\hat{\mu}_2, \hat{\nu}_2)$ is the one already given in our Eq.(45), and the new commuting pair is $(\hat{\mu}_1 = \hat{x}_1 - \hat{x}_4 - (\hat{x}_2 + \hat{x}_3), \hat{\nu}_1 = \hat{p}_1 - \hat{p}_4 + \hat{p}_2 + \hat{p}_3)$. With this new pair we can test bipartite entanglement in the bipartitions $\{\vec{\alpha}\}_1 = \{1|234, 2|134, 3|124, 4|123, 14|23\}$, with $|\gamma_{1,1|234}| = |\gamma_{1,2|134}| = |\gamma_{1,3|124}| = |\gamma_{1,4|123}| = 2$ and $|\gamma_{1,14|23}| = 4$ respectively, so $\gamma_{1,\min} = 2$ and $\tilde{\gamma}_{\min} = \min\{\gamma_{1,\min}, \gamma_{2,\min}\} = 2$. In this case we have $[\Theta] = [p_{1|234} + 2p_{2|134} + 2p_{3|124} + p_{4|123} + p_{12|34} + p_{13|24} + p_{14|23}] = 2$. We do not develop explicitly all cases but it is straightforward to verify that all genuine multipartite entanglement criteria presented in [36] can be recovered with the systematic presented in this work. The same is true for all the genuine 3-mode genuine entanglement criteria presented in [37]. For example, the genuine entanglement criteria in Eq.(5) of [37] is a special case of our genuine PPT-UNRLO entanglement criterion in Eq.(44) with $F = F_H$ (and $f(\tilde{\gamma}_{\min}) = f_H(\tilde{\gamma}_{\min}) = 1/4|\tilde{\gamma}_{\min}|^2$). In this case we only need two pairs of non-local observables:

$$\begin{aligned}\hat{\mu}_1 &= \hat{x}_1 - \hat{x}_2, & \hat{\nu}_1 &= \hat{p}_1 + \hat{p}_2 + \hat{p}_3, \\ \hat{\mu}_2 &= \hat{x}_1 - \hat{x}_3, & \hat{\nu}_2 &= \hat{p}_1 + \hat{p}_2 + \hat{p}_3.\end{aligned}\quad (46)$$

With the first pair $(\hat{\mu}_1, \hat{\nu}_1)$ we can test bipartite entanglement in the bipartitions $\{\vec{\alpha}\}_1 = \{1|23, 2|13\}$, and with the second pair in the bipartitions $\{\vec{\alpha}\}_2 = \{1|23, 3|12\}$, where $|\gamma_{m,\vec{\alpha}}| = 2 = \gamma_{m,\min}$ ($m = 1, 2$) for all the bipartitions, and therefore $\tilde{\gamma}_{\min} = 2$. Also, $[\Theta] = [2p_{1|23} + p_{2|13} + p_{3|12}] = 1$.

Our systematic approach also unveils many alternatives to construct genuine entanglement criteria for the same set of commuting non-local observables. This comes through the possibility to use any UR relation of the form in Eq.(2) valid for the associated set of mirrored non-local observables $\{(\hat{u}_{m,l}, \hat{v}_{m,l})\}$. In this regard it is worth noting that it is always possible to choose the set $\{(\hat{\mu}_m, \hat{\nu}_m)\}$ ($m = 1, \dots, M \leq L$) in such a way that, for every pair $(\hat{\mu}_m, \hat{\nu}_m)$, the associated set $\{(\hat{u}_{m,l}, \hat{v}_{m,l})\}$ consists of conjugate pairs. So, we can also use in our entanglement criterion in Eq.(44), any UR of the form in Eq.(2) that are valid only for conjugate pairs of observables, such as those involving the Rényi entropy Eq.(12). It is instructive to develop an example. In a 4-mode sys-

tem, we can use the set of observables:

$$\begin{aligned}\hat{\mu}_1 &= \frac{\hat{x}_1 - \hat{x}_2}{\sqrt{2}}, & \hat{\nu}_1 &= \frac{\hat{p}_1 + \hat{p}_2}{\sqrt{2}}, \\ \hat{\mu}_2 &= \frac{\hat{x}_2 - \hat{x}_3}{\sqrt{2}}, & \hat{\nu}_2 &= \frac{\hat{p}_2 + \hat{p}_3}{\sqrt{2}}, \\ \hat{\mu}_3 &= \frac{\hat{x}_1 - \hat{x}_3}{\sqrt{2}}, & \hat{\nu}_3 &= \frac{\hat{p}_1 + \hat{p}_3}{\sqrt{2}}, \\ \hat{\mu}_4 &= \frac{\hat{x}_3 - \hat{x}_4}{\sqrt{2}}, & \hat{\nu}_4 &= \frac{\hat{p}_3 + \hat{p}_4}{\sqrt{2}}, \\ \hat{\mu}_5 &= \frac{\hat{x}_2 - \hat{x}_4}{\sqrt{2}}, & \hat{\nu}_5 &= \frac{\hat{p}_2 + \hat{p}_4}{\sqrt{2}}, \\ \hat{\mu}_6 &= \frac{\hat{x}_1 - \hat{x}_4}{\sqrt{2}}, & \hat{\nu}_6 &= \frac{\hat{p}_1 + \hat{p}_4}{\sqrt{2}}.\end{aligned}\quad (47)$$

The sets of bipartitions $\{\vec{\alpha}\}_m$ that these operators test for bipartite entanglement coincide with the sets of bipartitions that the operators in Eq.(45) test for bipartite entanglement. In this case, $|\gamma_{m,\vec{\alpha}}| = \gamma_{m,\min} = \tilde{\gamma}_{\min} = 1$ with $m = 1, \dots, 6$. The associated set of mirrored non-local observables $\{(\hat{u}_{m,l}, \hat{v}_{m,l})\}$, with $m = 1, \dots, 6$ and $l = 1, \dots, 4$, are all conjugate pairs. Indeed, according to Eq.(20), for each conjugate pair $(\hat{u}_{m,l}, \hat{v}_{m,l})$ we have associated matrices $\mathbb{M}_{x,\vec{t}=\vec{\alpha}}$ and $\mathbb{M}_{p,\vec{t}=\vec{\alpha}}$ whose first rows, for example, correspond to the coefficients of the operators $\hat{u}_{m,l}$ and $\hat{v}_{m,l}$ respectively. In this case, because $\hat{u}_{m,l}$ and $\hat{v}_{m,l}$ are conjugate pairs, the different matrices satisfy $\mathbb{M}_{x,\vec{t}} = \gamma_{m,\vec{\alpha}}(\mathbb{M}_{p,\vec{t}})$, as we can readily check.

VI. EXAMPLE

The utility of our technique can be better appreciated by an example. Let us consider the quadripartite state

$$\hat{\rho} = (1 - b) |\psi\rangle \langle \psi| + b |\text{vac}\rangle \langle \text{vac}|, \quad (48)$$

where $|\text{vac}\rangle$ is the four-mode vacuum state and the state

$$\begin{aligned}|\psi\rangle &= \iiint \frac{1}{\pi \sqrt{s^2 t^2}} e^{-\frac{(x_1 + \frac{x_2 + x_3}{\sqrt{2}})^2}{4s^2}} e^{-\frac{(\frac{x_3 - x_2}{\sqrt{2}} + x_4)^2}{4s^2}} \times \\ &e^{-\frac{(x_1 - \frac{x_2 + x_3}{\sqrt{2}})^2}{4t^2}} e^{-\frac{(\frac{x_3 - x_2}{\sqrt{2}} - x_4)^2}{4t^2}} |x_1\rangle |x_2\rangle |x_3\rangle |x_4\rangle\end{aligned}\quad (49)$$

can be produced by creating two-mode squeezed states (modes 1/2 and 3/4) and then combining modes 2 and 3 on a 50/50 beam splitter [46]. Here the variables s and t are related to the usual squeezing parameter r by $s = (e^r)/2$ and $t = (e^{-r})/2$, where $r \rightarrow \infty$ corresponds to infinite squeezing. The state ρ is an incoherent combination of the state $|\psi\rangle$ and the vacuum state. We will

probe the entanglement using the operators

$$\hat{\mu}_1 = \hat{x}_1 - \frac{\hat{x}_2 + \hat{x}_3}{\sqrt{2}}, \quad (50a)$$

$$\hat{\nu}_1 = \hat{p}_1 + \frac{\hat{p}_2 + \hat{p}_3}{\sqrt{2}}, \quad (50b)$$

$$\hat{\mu}_2 = \frac{\hat{x}_3 - \hat{x}_2}{\sqrt{2}} - \hat{x}_4, \quad (50c)$$

$$\hat{\nu}_2 = \frac{\hat{p}_3 - \hat{p}_2}{\sqrt{2}} + \hat{p}_4. \quad (50d)$$

The marginal probability distributions associated to these operators can either be measured directly or obtained as marginals of the probability distributions $P(x_1, x_2, x_3, x_4)$ and $P(p_1, p_2, p_3, p_4)$. Following Table II, there are seven possible bipartitions. One can see immediately that the operator pair $\hat{\mu}_1$ and $\hat{\nu}_1$ could detect entanglement in every possible bipartition, except in the bipartition 4|123, using the type of criterion in Eq.(43). Analogously, the pair $\hat{\mu}_2$ and $\hat{\nu}_2$ could detect entanglement in all the bipartitions except in the bipartition 1|234. Thus, following the procedure outlined in section VB, we can use criteria (44) with $M = 2$, and also with the knowledge that each possible bipartition appears at least once in the sum over all tested bipartitions in the argument of the Θ function. Thus, we have $[\Theta] = 1$. Furthermore, direct calculation of the commutators $\gamma_{m,\vec{\alpha}}$ for all tested bipartitions (parameterized by $\vec{\alpha}$) shows that $\gamma_{m,\vec{\alpha}} \geq 1$, so that $\tilde{\gamma}_{\min} = 1$. We can then test the three entanglement criteria provided by the set of inequalities (11) in the multipartite form given by Eq. (44). Specifically, we test the linear inequality:

$$\Delta\hat{\mu}_1^2 + \Delta\hat{\nu}_1^2 + \Delta\hat{\mu}_2^2 + \Delta\hat{\nu}_2^2 \geq 1, \quad (51)$$

the product inequality:

$$2\Delta\hat{\mu}_1\Delta\hat{\nu}_1 + 2\Delta\hat{\mu}_2\Delta\hat{\nu}_2 \geq 1, \quad (52)$$

and the entropic inequality:

$$h[P_{\hat{\mu}_1}] + h[P_{\hat{\nu}_1}] + h[P_{\hat{\mu}_2}] + h[P_{\hat{\nu}_2}] \geq \ln(\pi e). \quad (53)$$

Violation of any of these inequalities guarantees genuine quadripartite entanglement. Figure 1 shows the violation of these inequalities for the state (48) as a function of the mixing parameter b . Here we chose squeezing parameter $r = 2$. One can see that the entropic criteria detects entanglement in regions where both the linear variance and variance product criteria fail. We emphasise that if the joint distribution probabilities, $P(x_1, x_2, x_3, x_4)$ and $P(p_1, p_2, p_3, p_4)$, were experimentally sampled, we have the freedom to choose any set $\{(\hat{\mu}_m(x_1, x_2, x_3, x_4), \hat{\nu}_m(p_1, p_2, p_3, p_4))\}$ to test genuine multipartite entanglement, using any uncertainty relation in criterion (44) that depends only on the marginal distributions $P_{\hat{\mu}_m}$ and $P_{\hat{\nu}_m}$.

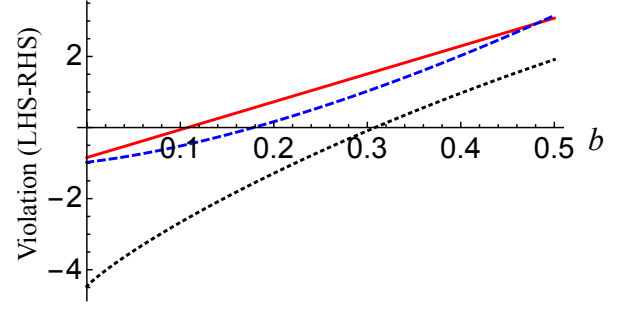


FIG. 1: (Color online.) Violation of three entanglement criteria for state (48) as a function of the mixing parameter b (both are dimensionless quantities). Genuine quadripartite entanglement is identified for negative values. The red solid curve corresponds to the linear criteria (51), the blue dashed line to the product criteria (52), and the black dotted line to the entropic criteria (53). For this non-gaussian state, the entropic criteria detects entanglement in regions where the variance criteria fail.

VII. FINAL REMARKS

The detection of genuine multipartite entanglement is a necessary and important step in the realization of quantum information tasks that exploit correlations between many parties. In the case of bipartite entanglement of continuous variable systems, the most widely adopted entanglement criteria are those based on constraints provided by uncertainty relations on non-local operators. Here we have provided a general framework to construct these types of criteria, based on the positive partial transpose criteria and uncertainty relations for non-local operators. Our criteria employ arbitrary uncertainty relations and consider bipartitions of generic size. We then use these results to build genuine multipartite entanglement criteria, and explicitly provide two categories. The first allows one to identify genuine multipartite entanglement with the measurement of a single pair of operators, and the second allows one to perform measurements on subsets of the constituent systems. These criteria are easily computable and experimentally friendly, in the sense that they require the reconstruction of a limited number of joint probability distributions. We expect our results to be useful in identifying genuine entanglement in a number of experimental systems.

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Appendix A

Here we are going to find the coefficients $(\mathbb{M}_{x,\bar{t}})_{k,j}$ and $(\mathbb{M}_{p,\bar{t}})_{k,j}$ that define the mirrored non-local operators in (20) such that the equality $F[\hat{\rho}^{T_{\bar{t}}}, P_{\hat{u}_{\bar{t}}}, P_{\hat{v}_{\bar{t}}}] = F[\hat{\rho}, P_{\hat{\mu}_{\bar{t}}}, P_{\hat{\nu}_{\bar{t}}}]$ in Eq.(15) holds for any functional. In order to do this, first note that the probability distributions $P_{\hat{\xi}}$ ($\xi = \mu, \nu$) are marginals:

$$P_{\hat{\xi}} = \int d\xi' \tilde{W}_{T_{\bar{t}}}(\mu_{\bar{t}}, \nu_{\bar{t}}), \quad (\text{A1})$$

(where $\hat{\xi} = \hat{\nu}$ if $\xi' = \nu$ and $\hat{\xi} = \hat{\mu}$ if $\xi' = \mu$) of the marginal distribution

$$\begin{aligned} \tilde{W}_{T_{\bar{t}}}(u_{\bar{t}}, v_{\bar{t}}) \equiv & \int \frac{d\mathbf{u}_{\bar{t}} d\mathbf{v}_{\bar{t}}}{|\gamma_{\bar{t}}|^n} W_{T_{\bar{t}}}(\mathbb{M}_{x,\bar{t}}^{-1} \mathbf{u}_{\bar{t}}, \mathbb{M}_{p,\bar{t}}^{-1} \mathbf{v}_{\bar{t}}) \times \\ & \times \delta(u_{k,\bar{t}} - u_{\bar{t}}) \delta(v_{k,\bar{t}} - v_{\bar{t}}) \end{aligned}$$

of the Wigner function $W_{T_{\bar{t}}}(\mathbf{x}, \mathbf{p})$ of the operator $\hat{\rho}^{T_{\bar{t}}}$. Now, we remember that [15]:

$$W_{T_{\bar{t}}}(\mathbf{x}, \mathbf{p}) = W(\mathbf{x}, \mathbb{A}_{\bar{t}} \mathbf{p}),$$

where W is the Wigner function of the original state $\hat{\rho}$ and $\mathbb{A}_{\bar{t}}$ is a diagonal matrix with ones in the location of modes that are not transposed and negative ones in the location of modes that are transposed. Making the change of variables $\boldsymbol{\mu}_{\bar{t}} = \mathbb{M}_{\mu,\bar{t}} \mathbf{u}_{\bar{t}}$ and $\boldsymbol{\nu}_{\bar{t}} = \mathbb{M}_{\nu,\bar{t}} \mathbf{v}_{\bar{t}}$ with the Jacobian equal to one we can write:

$$\begin{aligned} \tilde{W}_{T_{\bar{t}}}(u_{\bar{t}}, v_{\bar{t}}) = & \int \frac{d\boldsymbol{\mu}_{\bar{t}} d\boldsymbol{\nu}_{\bar{t}}}{|\gamma_{\bar{t}}|^n} W(\mathbb{M}_{x,\bar{t}}^{-1} \mathbb{M}_{\mu,\bar{t}}^{-1} \boldsymbol{\mu}_{\bar{t}}, \mathbb{A}_{\bar{t}} \mathbb{M}_{p,\bar{t}}^{-1} \mathbb{M}_{\nu,\bar{t}}^{-1} \boldsymbol{\nu}_{\bar{t}}) \times \\ & \times \delta\left(\sum_{i=1}^n (\mathbb{M}_{\mu,\bar{t}}^{-1})_{ki} \mu_{i,\bar{t}} - \mu_{\bar{t}}\right) \times \\ & \times \delta\left(\sum_{i=1}^n (\mathbb{M}_{\nu,\bar{t}}^{-1})_{ki} \nu_{i,\bar{t}} - \nu_{\bar{t}}\right) = \tilde{W}(\mu_{\bar{t}}, \nu_{\bar{t}}). \end{aligned} \quad (\text{A2})$$

In order for $\tilde{W}(\mu_{\bar{t}}, \nu_{\bar{t}})$ to be the marginal distribution of the Wigner function of the original state $\hat{\rho}$, associated with operators $\hat{\mu}_{\bar{t}}$ and $\hat{\nu}_{\bar{t}}$, the following conditions must be fulfilled: $\mathbb{M}_{x,\bar{t}}^{-1} \mathbb{M}_{\mu,\bar{t}}^{-1} = \mathbb{M}_{x,\bar{t}}^{-1}$ and $\mathbb{A}_{\bar{t}} \mathbb{M}_{p,\bar{t}}^{-1} \mathbb{M}_{\nu,\bar{t}}^{-1} = \mathbb{M}_{p,\bar{t}}^{-1}$. This is equivalent to:

$$\mathbb{M}_{\mu,\bar{t}} = \mathbb{1} \quad (\text{A3a})$$

$$\mathbb{M}_{\nu,\bar{t}} = \mathbb{M}_{p,\bar{t}} \mathbb{A}_{\bar{t}} \mathbb{M}_{p,\bar{t}}^{-1} = \mathbb{M}_{p,\bar{t}} \mathbb{A}_{\bar{t}} \mathbb{M}_{x,\bar{t}}^T \gamma_{\bar{t}}^{-1}, \quad (\text{A3b})$$

with $\det(\mathbb{M}_{\nu,\bar{t}}) = \det(\mathbb{A}_{\bar{t}}) = (-1)^{\tilde{n}}$ where $\tilde{n} = n_A$ for $\bar{t} = \bar{\alpha}$ and $\tilde{n} = n_B = n - n_A$ when $\bar{t} = \bar{\beta}$ (thus the Jacobian in the change of variables in Eq.(A2) were indeed

equal to one). Note that $\mathbb{M}_{\nu,\bar{t}}$ is an involutory matrix, *i.e.*, $\mathbb{M}_{\nu,\bar{t}}^2 = \mathbb{1}$ with signature \tilde{n} (the signature is the number of elements equal to -1 in $\mathbb{A}_{\bar{t}}$ [48]). If conditions in Eqs.(A3) are fulfilled then for the marginals $P_{\hat{\mu}_{\bar{t}}}$ and $P_{\hat{\nu}_{\bar{t}}}$ of $W(\mu_{\bar{t}}, \nu_{\bar{t}})$ we have

$$P_{\hat{u}_{\bar{t}}} = P_{\hat{\mu}_{\bar{t}}} \quad \text{and} \quad P_{\hat{v}_{\bar{t}}} = P_{\hat{\nu}_{\bar{t}}}, \quad (\text{A4})$$

where $P_{\hat{u}_{\bar{t}}}$ and $P_{\hat{v}_{\bar{t}}}$ are the marginals of $\tilde{W}_{T_{\bar{t}}}(u_{\bar{t}}, v_{\bar{t}})$.

The involutory property of $\mathbb{M}_{\nu,\bar{t}}$ (*i.e.* $\mathbb{M}_{\nu,\bar{t}} = \mathbb{M}_{\nu,\bar{t}}^{-1}$) and the Eqs.(A3) and (21) allows us to recognise in (A2) the operators $\hat{\mu}_{\bar{t}}$ and $\hat{\nu}_{\bar{t}}$ in Eq.(16) as:

$$\hat{\mu}_{\bar{t}} \equiv \hat{\mu}_{k,\bar{t}} = \sum_{j=1}^n \gamma_{\bar{t}} ((\mathbb{M}_{p,\bar{t}}^{-1})^T)_{k,j} \hat{x}_j \quad (\text{A5a})$$

$$\hat{\nu}_{\bar{t}} \equiv \hat{\nu}_{k,\bar{t}} = \sum_{j=1}^n (\mathbb{M}_{p,\bar{t}} \mathbb{A}_{\bar{t}})_{k,j} \hat{p}_j, \quad (\text{A5b})$$

where we use that $\mathbb{M}_{x,\bar{t}} = \gamma_{\bar{t}} (\mathbb{M}_{p,\bar{t}}^{-1})^T$. Therefore, comparing with Eqs.(16) we arrive at

$$\frac{h_j}{\gamma_{\bar{t}}} = ((\mathbb{M}_{p,\bar{t}}^{-1})^T)_{k,j} \quad (\text{A6a})$$

$$g_j = (\mathbb{M}_{p,\bar{t}} \mathbb{A}_{\bar{t}})_{k,j}, \quad (\text{A6b})$$

with $[\hat{\mu}_{\bar{t}}, \hat{\nu}_{\bar{t}}] = i(\mathbb{M}_{p,\bar{t}} \mathbb{A}_{\bar{t}} \mathbb{M}_{x,\bar{t}}^T)_{kk} = i \sum_{j=1}^n h_j g_j = i \delta_{\bar{t}} \mathbb{1}$, coinciding with the result in Eq.(17). Because the matrix $\mathbb{M}_{\mu,\bar{t}} = \mathbb{1}$, the original and mirrored non-local position-type observable coincide, *i.e.* $\hat{u}_{\bar{t}} = \hat{\mu}_{\bar{t}}$. The coefficient of the mirrored non-local observable $\hat{v}_{\bar{t}}$ in Eq.(20) can be obtained from Eq.(A6b) giving the result in Eq.(23). Therefore, the commutator between the mirrored observables is $[\hat{u}_{\bar{t}}, \hat{v}_{\bar{t}}] = i \gamma_{\bar{t}} \mathbb{1}$, with $\gamma_{\bar{t}}$ given in Eq.(25).

In this way, we see that the coefficients of the mirrored operators $\hat{u}_{\bar{t}}$ and $\hat{v}_{\bar{t}}$ are determined once we specify the p-matrix of the bipartition $\mathbb{M}_{p,\bar{t}}$, whose k^{th} row is given in Eq.(23) and with the k^{th} row of $(\mathbb{M}_{p,\bar{t}}^{-1})^T$ given in Eq.(A6a). Without loss of generality we can choose $k = 1$. In Appendix B we give the general structure of a matrix $\mathbb{M}_{p,\bar{t}}$ with these properties. This proves the equality $F[\hat{\rho}^{T_{\bar{t}}}, P_{\hat{u}_{\bar{t}}}, P_{\hat{v}_{\bar{t}}}] = F[\hat{\rho}, P_{\hat{\mu}_{\bar{t}}}, P_{\hat{\nu}_{\bar{t}}}]$ in Eq.(15).

Appendix B

Here we give the general structure of a $n \times n$ real matrix $\mathbb{M}_{p,\bar{t}}$ that satisfy Eqs.(23) and (A6a) for a given value of $\gamma_{\bar{t}}$. Without loss of generality we choose $k = 1$ so:

$$(\mathbb{M}_{p,\vec{t}})_{ij} = \begin{cases} \bar{g}_1 = \bar{g}_1(\gamma_{\vec{t}}, \delta_{\vec{t}}, \bar{g}_2, \dots, \bar{g}_n, h_1, \dots, h_n) & \text{for } i = j = 1 \\ \bar{g}_j & \text{for } i = 1 \text{ and } 1 < j < n \\ \bar{g}_n = \bar{g}_n(\gamma_{\vec{t}}, \delta_{\vec{t}}, \bar{g}_2, \dots, \bar{g}_n, h_1, \dots, h_n) & \text{for } i = 1 \text{ and } j = n \\ Q_{ij} & 1 < i \leq n \text{ and } 1 \leq j \leq n-1 \\ -\frac{1}{h_n} \left(\sum_{l=1}^{n-1} h_l Q_{il} \right) & \text{for } 1 < i \leq n \text{ and } j = n \end{cases} \quad (\text{B1})$$

where $\bar{g}_j = -g_j$ if j is one component of the vector \vec{t} or $\bar{g}_j = g_j$ otherwise ($\vec{t} = \vec{\alpha}$ or $\vec{t} = \vec{\beta}$), \bar{g}_1 and \bar{g}_n are solutions of the equations:

$$\sum_{j=1}^n \bar{g}_j h_j = \gamma_{\vec{t}} \quad (\text{B2})$$

$$\sum_{j=1}^n g_j h_j = \delta_{\vec{t}} \quad (\text{B3})$$

and the matrix elements Q_{ij} ($1 < i \leq n$ and $1 \leq j \leq n-1$) are arbitrary. In the case of a “seed” partition defined by the vector $\vec{t} = \vec{s} = (1, \dots, n_A)$ the explicit form of $\mathbb{M}_{p,\vec{s}}$ is:

$$(\mathbb{M}_{p,\vec{s}})_{ij} = \begin{cases} -g_1 = \frac{-1}{h_1} \left(\frac{\delta_{\vec{s}} - \gamma_{\vec{s}}}{2} - \sum_{l=2}^{n_A} g_j h_l \right) & \text{for } i = j = 1 \\ -g_j & \text{for } i = 1 \text{ and } 1 \leq j \leq n_A \\ g_j & \text{for } i = 1 \text{ and } n_A < j < n \\ g_n = \frac{1}{h_n} \left(\frac{\delta_{\vec{s}} + \gamma_{\vec{s}}}{2} - \sum_{l=n_A+1}^{n-1} g_j h_l \right) & \text{for } i = 1 \text{ and } j = n \\ Q_{ij} & 1 < i \leq n \text{ and } 1 \leq j \leq n-1 \\ -\frac{1}{h_n} \left(\sum_{l=1}^{n-1} h_l Q_{il} \right) & \text{for } 1 < i \leq n \text{ and } j = n \end{cases} \quad (\text{B4})$$

Appendix C

Let's see the concavity of the functional $F_E[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] = h[P_{\hat{\mu}}] + h[P_{\hat{\nu}}]$, i.e. if $\hat{\rho} = \sum_j p_j \hat{\rho}_j$ ($\sum_j p_j = 1$) then $F_E[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] \geq \sum_j p_j F_E[\hat{\rho}_j, P_{\hat{\mu}}, P_{\hat{\nu}}]$. The Wigner function of an arbitrary n -mode state $\hat{\rho}$ is $W(\mathbf{x}, \mathbf{p}) = W(\mathbb{M}_{x,\vec{t}}^{-1} \boldsymbol{\mu}_{\vec{t}}, \mathbb{M}_{p,\vec{t}}^{-1} \boldsymbol{\nu}_{\vec{t}})$, so the functional F_E can be determined from the marginal distribution (see Eq.(A2)): $\tilde{W}(\mu, \nu) = \int \frac{d\boldsymbol{\mu}_{\vec{t}} d\boldsymbol{\nu}_{\vec{t}}}{|\gamma_{\vec{t}}|^n} W(\mathbb{M}_{x,\vec{t}}^{-1} \boldsymbol{\mu}_{\vec{t}}, \mathbb{M}_{p,\vec{t}}^{-1} \boldsymbol{\nu}_{\vec{t}})$. If we set $\hat{\rho} = \sum_j p_j \hat{\rho}_j$ ($\sum_j p_j = 1$), the Wigner function of $\hat{\rho}$ is the convex sum of the Wigner functions of the states $\hat{\rho}_j$, i.e. $W(\mathbf{x}, \mathbf{p}) = \sum_j p_j W_j(\mathbf{x}, \mathbf{p})$, and therefore for the marginal distribution we have:

$$\tilde{W}(\mu, \nu) = \sum_j p_j \tilde{W}_j(\mu, \nu). \quad (\text{C1})$$

Because $P_{\hat{\xi}}(\xi) = \int d\xi' \tilde{W}(\mu, \nu)$ (where $\hat{\xi} = \hat{\nu}$ if $\xi' = \nu$ and $\hat{\xi} = \hat{\mu}$ if $\xi' = \mu$), from Eq.(C1) we immediately obtain:

$$P_{\hat{\xi}}(\xi) = \sum_j p_j P_{j,\hat{\xi}}(\xi). \quad (\text{C2})$$

Now, we use that the Shannon entropy: $G[P(\xi)] \equiv -\int_{-\infty}^{\infty} d\xi P(\xi) \ln(P(\xi))$ is a strictly concave functional of $P(\xi)$, i.e. if $P(\xi) = \sum_j p_j P_j(\xi)$ then $G[P(\xi)] \geq \sum_j p_j G[P_{j,\hat{\xi}}(\xi)]$ (see below), so we can write,

$$\begin{aligned} F_E[\hat{\rho}, P_{\hat{\mu}}, P_{\hat{\nu}}] &= G[P_{\hat{\mu}}(\mu)] + G[P_{\hat{\nu}}(\nu)] \geq \\ &\geq \sum_j p_j (G[P_{j,\hat{\mu}}(\mu)] + G[P_{j,\hat{\nu}}(\nu)]) = \\ &= \sum_j p_j F_E[\hat{\rho}_j, P_{\hat{\mu}}, P_{\hat{\nu}}]. \end{aligned} \quad (\text{C3})$$

We can see that $G[P(\xi)]$ is a strictly concave functional in the following way. First, we note that $P(\xi) = \sum_j p_j P_j(\xi) \geq P_j(\xi)$, thus because $-\ln(x)$ is a strictly crescent function we also have $-\ln(P(\xi)) > -\ln(P_m(\xi))$.

Therefore, we immediately have:

$$\begin{aligned}
 G[P(\xi)] &= - \int_S d\xi \left(\sum_j p_j P_j(\xi) \right) \ln(P(\xi)) \geq \\
 &\geq \sum_j p_j \left(- \int_S d\xi P_j(\xi) \ln(P_j(\xi)) \right) = \\
 &= \sum_j p_j G[P_j(\xi)]. \tag{C4}
 \end{aligned}$$

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